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STRUCTURAL BEHAVIOR OF COMPOSITE MATERIALS

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SUMMARY

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This report covers the period from May 14 to August 14, 1963. Work was concentrated on the theoretical prediction of the mechanical constitutive equation of laminated anisotropic plates, with special emphasis on cross-ply and angle-ply composites. The material coefficients are expressed as functions of lamination parameters. Through the week ending August 9, 1963, 2625 man-hours had been expended; costs that date were $44,605. author

ACKNOWLEDGEMENT

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SECTION 1

THEORY OF LAMINATED PLATES

1.1 INTRODUCTION

The laminated plate under present investigation consists of n plies of homogeneous anisotropic sheets. The stress-strain relation of the k-th ply is:

$$\epsilon_i^{(k)} = S_{ij}^{(k)} \sigma_j^{(k)} \tag{1}$$

$$\sigma_i^{(k)} = C_{ij}^{(k)} \epsilon_j^{(k)} \tag{2}$$

where $1 \leq k \leq n$; ϵ_i = strain components; σ_i = stress components; S_{ij} = compliance matrix; C_{ij} = stiffness matrix; i, j = 1, 2, 6; repeated indices represent summation.

In the classical plate theory, the variables used are:

$$N_i = \text{stress resultant} = \int_{-h/2}^{h/2} \sigma_i \, d\, z$$

$$M_i = \text{stress couple} = \int_{-h/2}^{h/2} \sigma_i \, z \, d z$$

$$\epsilon_i^o = \text{middle-plane strain}$$

$$\kappa_i = \text{bending curvature}$$

where the total strain $\epsilon_i = \epsilon_i^o + z \kappa_i$. Thus the constitutive equation of a laminated anisotropic plate, in matrix form, is

$$\begin{bmatrix} N \\ \\ M \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right] \begin{bmatrix} \epsilon^o \\ \\ \kappa \end{bmatrix} \tag{3}$$

where the composite material matrix is partitioned into four sub-matrices so that:

$$\begin{aligned} [A], \quad [B], \quad [D] &= A_{ij}, B_{ij}, D_{ij} \\ &= \int_{-h/2}^{h/2} (1, z, z^2) \, C_{ij} \, d z \end{aligned} \tag{4}$$

The purpose of the present investigation is to study the nature of A, B, and D matrices as functions of material and lamination parameters. The material parameters refer to the C_{ij} matrix of the unit plies; the lamination parameters to the thickness and orientation of each ply and the total number and the stacking sequence of all the plies.

1.2 INVERSION OF COMPOSITE MATRIX

It is often more convenient to use the inverted constitutive equation of Equation (3). This can be easily accomplished, as follows: Equation (3) can be written as:

$$N = A \epsilon^{o} + B \kappa \tag{5}$$

$$M = B \epsilon^{o} + D \kappa \tag{6}$$

From (5) $\epsilon^{o} = A^{-1} N - A^{-1} B \kappa$ (7)

Substituting Equation (7) into (6) and rearranging:

$$M = B A^{-1} N + (-B A^{-1} B + D) \kappa \tag{8}$$

Combining Equations (7) and (8), in matrix form:

$$\begin{bmatrix} \epsilon^{o} \\ M \end{bmatrix} = \left[\begin{array}{c|c} A^{-1} & -A^{-1} B \\ \hline B A^{-1} & D - B A^{-1} B \end{array}\right] \begin{bmatrix} N \\ \kappa \end{bmatrix} \tag{9}$$

$$= \left[\begin{array}{c|c} A^{*} & B^{*} \\ \hline C^{*} & D^{*} \end{array}\right] \begin{bmatrix} N \\ \kappa \end{bmatrix} \tag{10}$$

where the definitions of the star matrices are self-evident. This equation is a partial inversion of Equation (3). The components of the star matrices are used as the coefficients of the differential equation of equilibrium for laminated plates and shells. Rewriting Equation (10)

$$\epsilon^o = A^* N + B^* \kappa \tag{11}$$

$$M = C^* N + D^* \kappa \tag{12}$$

F ~m Equation (12),

$$\kappa = D^{*^{-1}} M - D^{*^{-1}} C^* N \tag{13}$$

Substituting Equation (13) into (12)

$$\epsilon^o = B^* D^{*^{-1}} M + (A^* - B^* D^{*^{-1}} C^*) N \tag{14}$$

Combining Equation (13) and (14), in matrix form

$$\begin{bmatrix} \epsilon^o \\ \kappa \end{bmatrix} = \left[\begin{array}{c:c} A^* - B^* D^{*^{-1}} C^* & B^* D^{*^{-1}} \\ \hdashline -D^{*^{-1}} C^* & D^{*^{-1}} \end{array}\right] \begin{bmatrix} N \\ M \end{bmatrix} \tag{15}$$

$$= \left[\begin{array}{c:c} A' & B' \\ \hdashline C' & D' \end{array}\right] \begin{bmatrix} N \\ M \end{bmatrix} \tag{16}$$

where the definitions of the prime matrices are self-evident. This equation is the complete inversion of Equation (3). Equation (16) is more convenient to use if the amount of stretching and bending is given for a given problem.

1.3 THE CONSTITUTIVE EQUATION

Equation (3), or its alternate form as shown in Equation (10) or (16), is the most general constitutive equation for laminated anisotropic plates and shells. Since C_{ij} is a fourth rank symmetric tensor, A_{ij}, B_{ij} and D_{ij} must retain the same tensorial properties of C_{ij}, i.e., they are also fourth rank symmetric tensors. As defined in Equation (4), A_{ij}, B_{ij} and D_{ij} are obtained by integration along the z-axis. This is a scalar operation, which, by definition, does not alter the tensorial property of C_{ij}. Thus, in general, there are 18 independent constants in the present constitutive equation.

If the plate is homogeneous, i.e., C_{ij} is not a function of z, then

$$A_{ij} = \frac{12}{h^2} D_{ij}, \qquad B_{ij} = 0 \tag{17}$$

The only independent matrix is A, thus the number of independent constants is at most six.

If he laminated plate consists of isotropic plies only, i.e., it is a generalized sandwich plate, then

$$\begin{aligned} C_{11} &= C_{22} \\ C_{66} &= (C_{11} - C_{12}) / 2 \\ C_{16} &= C_{26} = 0 \end{aligned} \tag{18}$$

There are only two independent constants for each unit ply, instead of generally six constants. Thus, for a laminated isotropic plate, A_{ij}, B_{ij}, D_{ij} each can have at most two independent constants, making a total of six constants. These six constants are, however, different from the six for homogeneous anisotropic plates shown in Equation (17).

If the plate is homogeneous and isotropic, the combined conditions of Equation (17) and (18) reduce the number of independent constants to two.

If C_{ij} is an even function, i.e., C_{ij} is symmetrical with respect to the $z = 0$ plane, B_{ij} is identically zero. The number of independent constants for this type of laminated anisotropic plate is reduced from 18 to 12. Since C_{ij} cannot in general be an odd function, A_{ij} and D_{ij} cannot be identically zero, with the possible exception of the 12, 16 and 26 components of the A_{ij} and D_{ij} matrices.

The number of independent constants for a laminated anisotropic plate is affected by the elastic symmetry of the C_{ij} for each unit ply. A general discussion of this subject is too lengthy for the present purpose. Only specific types of laminated plates will be covered, i.e., cross-plied and angle-plied plates. It should now be pointed out that in the process of lamination, the elastic symmetry of the original C_{ij}, e.g., orthotropic, angular, cubic, isotropic symmetries, in general, is not carried directly into the laminated plate. The level of symmetry may be increased or decreased depending on the type of lamination. For a given laminated plate, the elastic symmetries of A_{ij}, B_{ij}, and D_{ij} need not be the same.

SECTION 2

CROSS-PLY COMPOSITES

2.1 LAMINATION PARAMETERS

The cross-ply composite consists of n layers of an orthotropic material stacked with alternating orientation of 90° between layers. The principal direction of the odd layer coincides with the x-axis, and the even layers with the y-axis. All the odd layers have the same thickness. The even layers also have the same thickness, which may be different from that of the odd layers. Another lamination parameter m is defined as the ratio between the total thickness of the odd layers over the total thickness of the even layers.

The purpose of this section is to determine the composite material matrices A, B and D as functions of the material parameter C_{ij} and lamination parameters m and n.

Assuming each unit layer is homogeneous, the integrations of Equation (3) can be replaced by summations, as follows:

$$A_{ij} = \sum_{k=1}^{n} C_{ij}^{(k)} (h_{k+1} - h_k) \qquad (19)$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} C_{ij}^{(k)} (h^2_{k+1} - h_k^2) \tag{20}$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} C_{ij}^{(k)} (h^3_{k+1} - h_k^3) \tag{21}$$

For cross-ply composites, all layers are orthotropic, so that components of the C_{ij} for odd layers are:

$$C_{11}, C_{22}, C_{12}, C_{66} \text{ with } C_{16} = C_{26} = 0$$

The components of the C_{ij} for even layers are the same as those for the odd layers except C_{11} and C_{22} are interchanged.

2.2 DERIVATION OF A, B, D MATRICES

The summations of Equations (19), (20) and (21) can be expressed in closed form for the cross-ply composite. This is accomplished by taking advantage of the properties of series. In a straight-forward but laborious manner one can derive the following, where F = ratio of principal stiffnesses of the unit ply = $C_{22} / C_{11} = E_{22} / E_{11}$:

i. <u>For n Odd</u>

$$A_{11} = \frac{1}{1 + m} (m + F) h C_{11} \tag{22}$$

$$A_{22} = \frac{1}{1 + M} \quad (1 + m F) \quad h C_{11} = \frac{1 + m F}{m + F} \quad A_{11} \tag{23}$$

$$A_{12} = h C_{12} \tag{24}$$

$$A_{66} = h C_{66} \tag{25}$$

$$A_{16} = A_{26} = 0 \tag{26}$$

$$B_{ij} = 0 \tag{27}$$

$$D_{11} = \left[(F - 1) P + 1\right] \quad \frac{h^3}{12} \quad C_{11} \tag{28}$$

$$= \left[(F - 1) P + 1\right] \quad \frac{1 + m}{m + F} \quad \frac{h^2}{12} \quad A_{11} \tag{29}$$

$$D_{22} = \left[(1 - F) P + F\right] \frac{h^3}{12} \quad C_{11} \tag{30}$$

$$= \left[(1 - F) P + F\right] \frac{1 + m}{m + F} \quad \frac{h^2}{12} \quad A_{11} \tag{31}$$

where $$P = \frac{1}{(1 + m)^3} + \frac{m\ (n - 3) \left[m\ (n - 1) + 2\ (n + 1)\right]}{(n^2 - 1) \quad (1 + m)^3}$$

$$D_{12} = \frac{h^3}{12} \quad C_{12} \tag{32}$$

$$D_{66} = \frac{h^3}{12} \quad C_{66} \tag{33}$$

$$D_{16} = D_{26} = 0 \tag{34}$$

ii. For n Even

Same as the n odd case except for the following components:

$$B_{11} = -B_{22} = \frac{m\ (F - 1)}{n\ (1 + m)^2}\ h^2\ C_{11} = \frac{m\ (F - 1)}{n\ (1 + m)\ (m + F)}\ h\ A_{11} \tag{35}$$

$$D_{11} = \left[(F - 1)\ Q + 1 \right]\ \frac{h^3}{12}\ C_{11} \tag{36}$$

$$= \left[(F - 1)\ Q + 1 \right]\ \frac{h^2}{12}\ \frac{1 + m}{m + F}\ A_{11} \tag{37}$$

$$D_{22} = \left[(1 - F)\ Q + F \right]\ \frac{h^3}{12}\ C_{11} \tag{38}$$

$$= \left[(1 - F)\ Q + F \right]\ \frac{h^2}{12}\ \frac{1 + m}{m + F}\ A_{11} \tag{39}$$

where $Q = \frac{1}{1 + m} + \frac{8\ m\ (m - 1)}{n^2\ (1 + m)^3}$

2.3 DISCUSSIONS OF A, B, D MATRICES

Using Equations (22) and (23), A_{11} and A_{22} are plotted, in dimensionless form, in Figure 1. The other components of A_{ij} are not plotted because either they are identically zero or remain constant, as shown in Equations (24), (25) and (26). One can conclude that for cross-ply construction:

1) A_{ij} remains orthotropic;
2) A_{ij} is independent of n, the total number of plies;

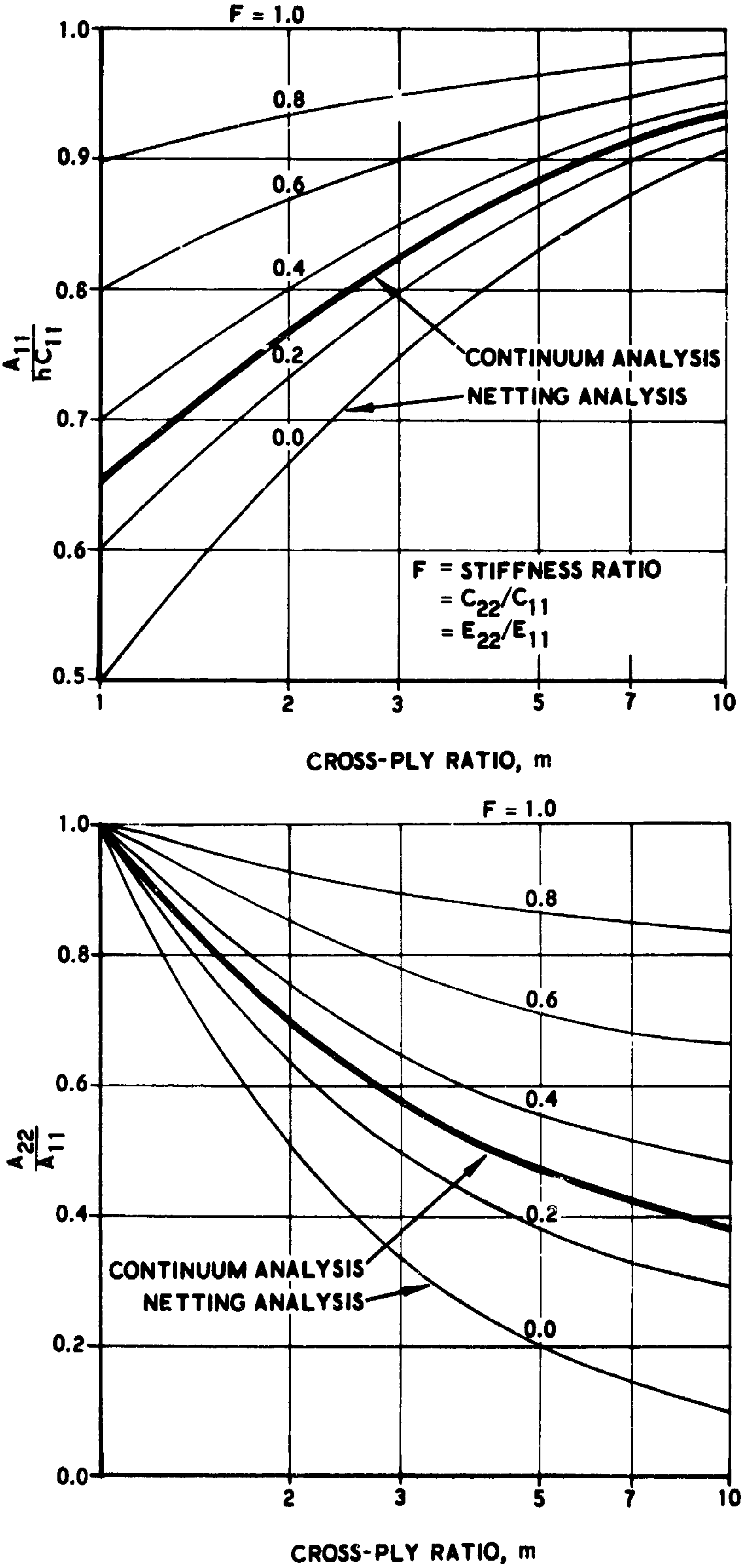

FIGURE 1. DIMENSIONLESS STIFFNESS COMPONENTS A_{11} AND A_{22}

3) A_{11} and A_{22} are affected drastically by both the stiffness ratio F and the cross-ply ratio m.

4) The average stiffness ratio for filament-wound unidirectional ply is approximately 0.3 for resin contents by weight between 15 to 30 percent. This is also plotted in Figure 1. As a comparison, the stiffness based on netting analysis, which assumes $C_{22} = 0$ or $F = 0$, is also shown. The difference between the two cases, the former case being called the continuum analysis for identification and the latter the netting analysis, is quite substantial. The cross-ply ratio, m, for "balanced design" based on netting analysis is equal to 2.0.

The B_{ij} is identically zero for cross-ply construction except for B_{11}, which is equal to $-B_{22}$, when n is even. Using Equation (35), B_{11} and B_{22} are plotted, in dimensionless form, in Figure 2. The physical significance of B_{11} can be interpreted as a measure of the shifting of the neutral plane. The numerical value in Figure 2 represents the amount of shifting as a fraction of the total plate thickness. The maximum amount of shifting occurs when $n = 2$. The shifting is inversely proportionate with n. Hence, it becomes small for a large number of plies. Again the difference between the netting and continuum analyses is very significant. It is interesting to observe that this B_{ij} has only one independent component, i.e., B_{11} with $B_{22} = -B_{11}$ and $B_{12} = B_{66} = B_{16} = B_{26} = 0$. This matrix is more than orthotropic in the sense that its level of elastic symmetry is higher than the orthotropic case. The transformation property of the B_{ij} is also shown in Figure 2. $B_{11} = -B_{22}$ holds for all angles. B_{12} and B_{66} remain identically zero. $B_{16} = B_{26}$ also holds for all angles. At 45°, $B_{11} = B_{22} = 0$, i.e., the shifting of the neutral plane is zero. At this orientation, the cross-ply becomes the same as an angle-ply, for which, the neutral plane does not shift.

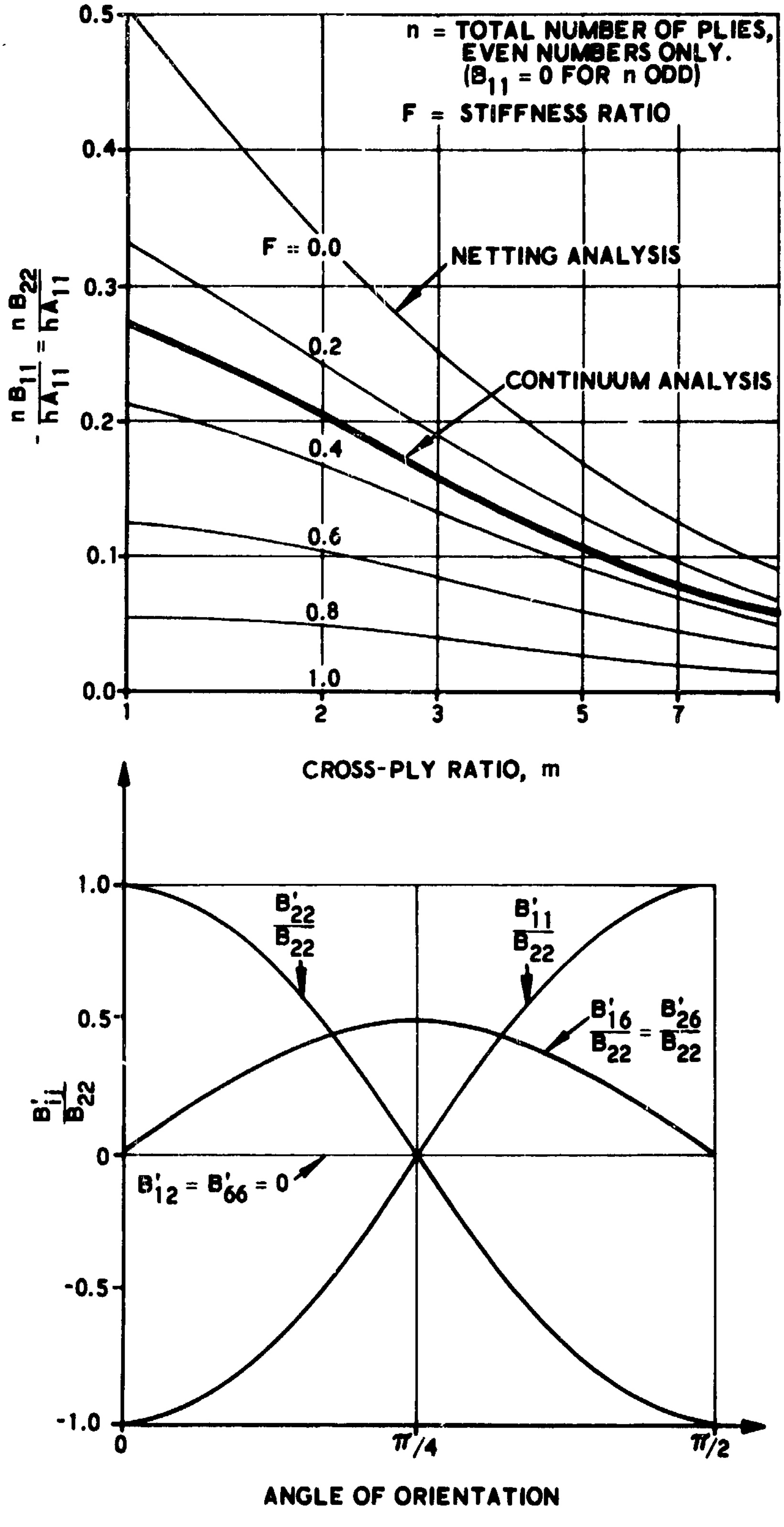

FIGURE 2. DIMENSIONLESS COUPLING TERM B_{11} AND ITS TRANSFORMATION PROPERTY

The D matrix is much more complicated than A and B above. Since D_{11} and D_{22} depend on both the total number of plies n and stiffness ratio F, only a few combinations of n and F are shown in Figure 3. Again, F = 0.0 represent the netting analysis and F = 0.3 the continuum analysis. First of all, for cross-ply composites D_{ij} is orthotropic. D_{11} and D_{22} approach $h^2 A_{11} / 12$ and $h^2 A_{22} / 12$, respectively, i.e., the cross-ply composite approaches a homogeneous plate, when:

1) m becomes large;
2) n becomes large; or
3) F becomes 1.

From Figure 3, it is clear that the most expedient method of making a laminated plate behave as a homogeneous plate within 10 percent accuracy is to use four plies for practically all values of m and F.

Substantial difference between the continuum and netting analyses is illustrated in Figure 3. For a given cross-ply ratio, say m = 2, the dimensionless flexural rigidities vary significantly depending on the number of plies, with n = 3 and 2 as the upper and lower bounds.

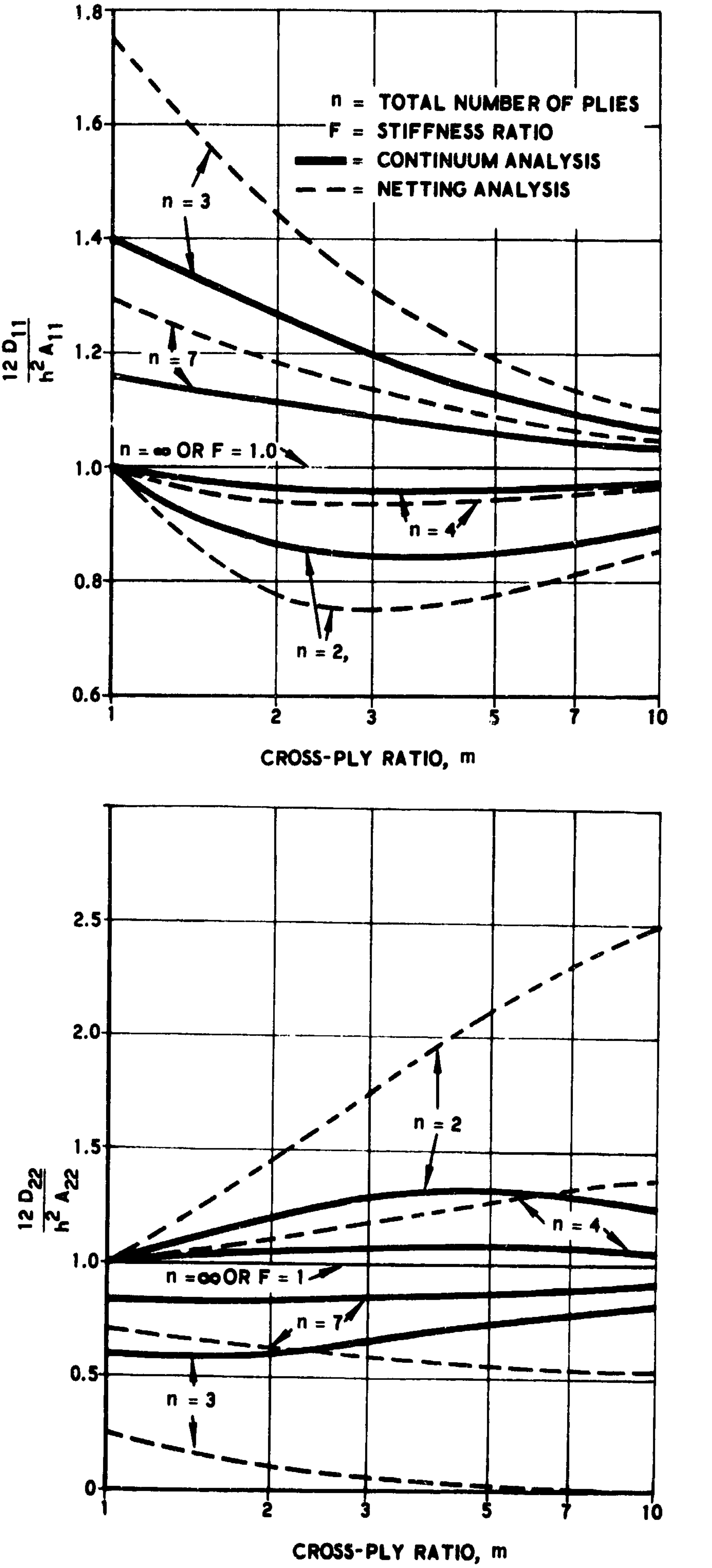

FIGURE 3. DIMENSIONLESS FLEXURAL RIGIDITIES D_{11} AND D_{22}

SECTION 3

ANGLE-PLY COMPOSITES

3.1 LAMINATION PARAMETERS

The angle-ply composite consists of n unit plies of an orthotropic material with an alternating angle of orientation between layers. The odd plies are orientated with an angle + θ from the x-axis and the even plies - θ. All plies have the same thickness. The lamination parameters for the angle-ply composite are the total number of plies n and the lamination angle θ.

The purpose of this section is to determine the composite material matrices A, B, D as functions of the material parameter C_{ij} of the unit ply and the lamination parameters n and θ.

3.2 DERIVATION OF A, B, D MATRICES

As stated in Section 2.2, the A, B, D matrices can be obtained by summations shown in Equations (19), (20) and (21). For angle-ply composites, these summations can be further simplified. In fact, A, B, D can be computed by very simple equations, which can be easily derived. These equations are derived by expanding the summations and using the conditions of the angle-ply composite (symmetric orientation of unit plies of equal thicknesses). The equations are shown in the following; where the C_{ij} is the stiffness matrix with + θ orientation:*

i. <u>For n Odd</u>

$$A_{11}, A_{22}, A_{12}, A_{66} = h\ (C_{11}, C_{22}, C_{66}, C_{12}) \tag{40}$$

$$(A_{16}, A_{26}) = \frac{h}{n}\ (C_{16}, C_{26}) \tag{41}$$

$$B_{ij} = 0 \tag{42}$$

$$(D_{11}, D_{22}, D_{12}, D_{66}) = \frac{h^3}{12}\ (C_{11}, C_{22}, C_{12}, C_{66}) \tag{43}$$

$$(D_{16}, D_{26}) = \frac{h^3}{12}\ \left(\frac{3\ n^2 - 2}{n^3} \right)\ (C_{16}, C_{26}) \tag{44}$$

ii. <u>For n Odd</u>

Same as the n even case except for the following components:

$$A_{16} = A_{26} = 0 \tag{45}$$

$$(B_{16}, B_{26}) = -\frac{h^2}{n}\ (C_{16}, C_{26}) \tag{46}$$

$$D_{16} = D_{26} = 0 \tag{47}$$

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*The C_{ij} for - θ orientation is equal to that of the + θ orientation except the sign for C_{16} and C_{26} is changed.

3.3 DISCUSSIONS OF A, B, D MATRICES

The absolute values of A_{11} for a representative filament-wound angle-ply composite is plotted against the lamination angle θ in Figure 4. The other components of A_{ij} are also shown in Figure 4 in dimensionless form. A_{11}, A_{22}, A_{12} and A_{66} are independent of the number of plies, n. A_{16} and A_{26}, however, are dependent on n. When n even, they are zero. When n odd, A_{16} and A_{26} are inversely proportionate with n. Thus, the maximum absolute values for A_{16} and A_{26} occur when n = 3. It is interesting to know that for n even A_{ij} is orthotropic, for n odd it is not orthotropic because the plane of elastic symmetry is destroyed. For the latter case, the number of independent constants is six. This is a truly anisotropic system, corresponding to the triclinic case for three dimensional bodies. Also shown in Figure 4 is the commonly used lamination angle of 53-3/4°, which is conceived as the optimum angle for internally pressured vessels using the netting analysis. From the standpoint of continuum analysis, there is no apparent reason to restrict the use of the lamination angle to one specific value.

The B matrix for angle-ply composites is identically zero for n odd, and components for n even. The dimensionless B_{16} is plotted in Figure 5. The significance of this ratio can be seen, as follows:

For uniaxial extension, the only non-zero component on the right-hand-side of Equation (3) is ϵ_1^o. Expanding Equation (3),

$$N_1 = A_{11} \epsilon_1^o$$

$$N_2 = A_{12} \epsilon_1^o$$

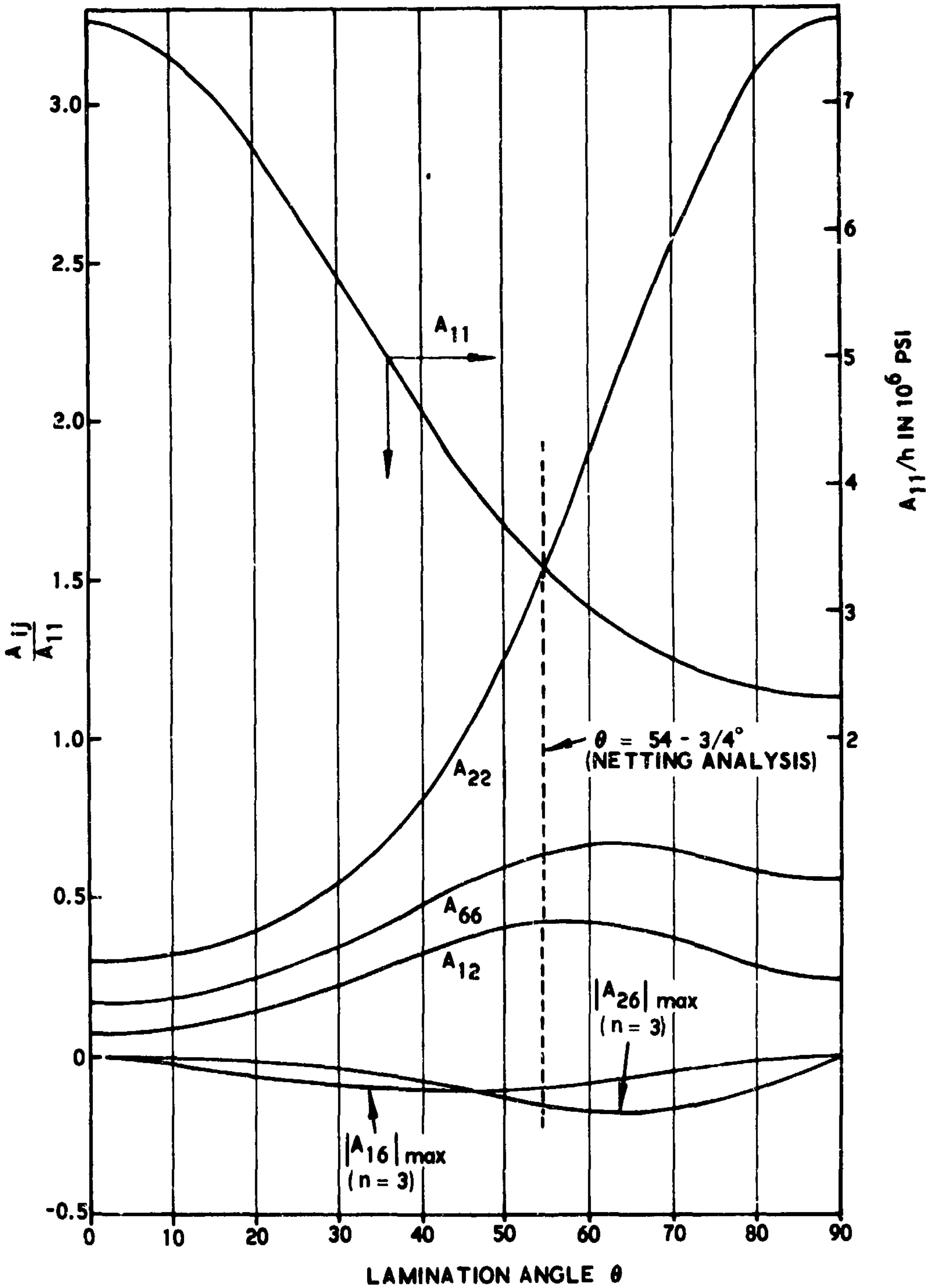

FIGURE 4. A_{11} AND DIMENSIONLESS A_{ij} FOR REPRESENTATIVE FILAMENT-WOUND ANGLE-PLY

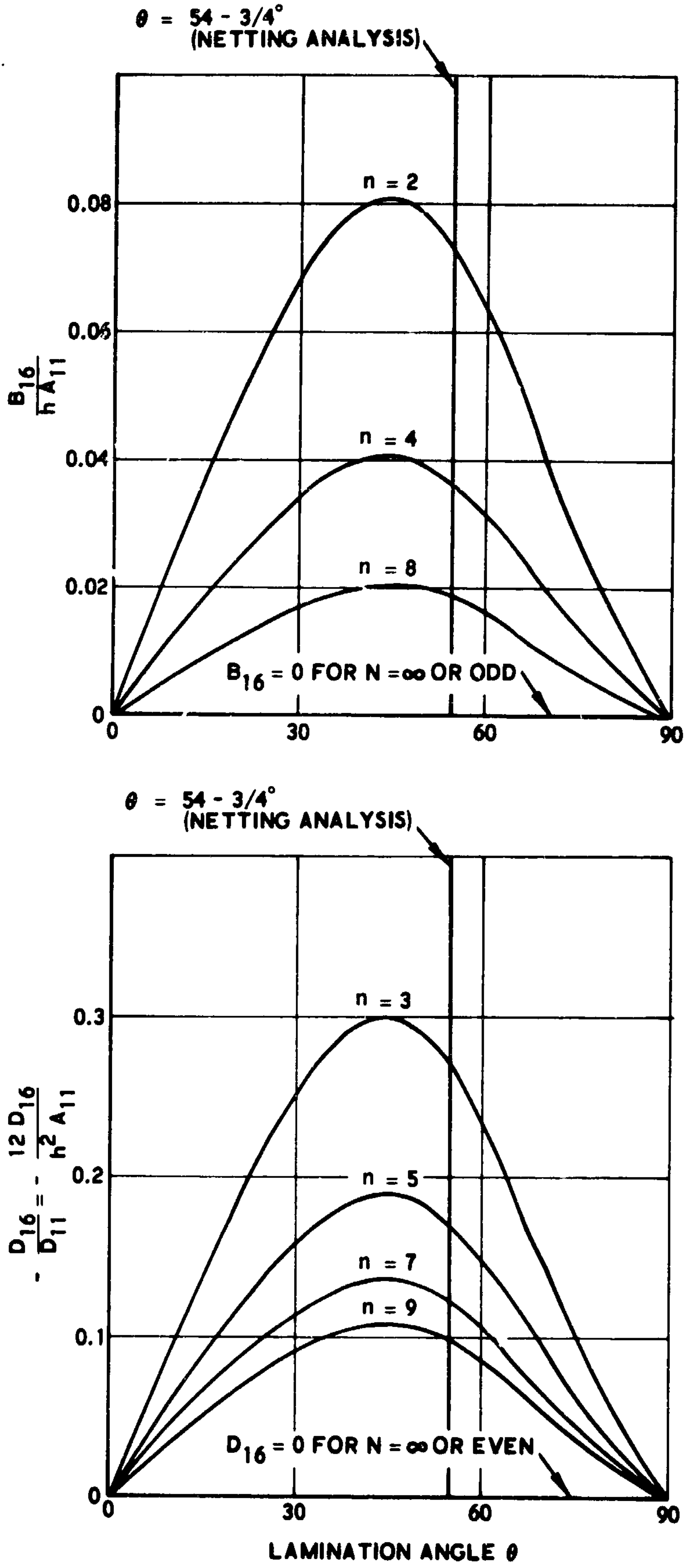

FIGURE 5. DIMENSIONLESS B_{16} AND D_{16} FOR REPRESENTATIVE FILAMENT-WOUND ANGLE-PLY

$$N_6 = A_{16} \epsilon_1^o = 0$$

$$M_1 = B_{11} \epsilon_1^o = 0 \tag{48}$$

$$M_2 = B_{12} \epsilon_1^o = 0$$

$$M_6 = B_{16} \epsilon_1^o$$

Thus

$$\frac{B_{16}}{h\ A_{11}} = \frac{M_6}{h\ N_1} \tag{49}$$

This ratio signifies the ratio of the induced twisting moment to the in-plane stress resultant. From this ratio one can then compute the ratio of the shear stress over the normal stress for the uniaxial extension. Similarly, one can show

$$\frac{B_{26}}{h\ A_{22}} = \frac{M_6}{h\ N_2} \tag{50}$$

This latter ratio has the same numerical value as Equation (49) except the complement of the lamination angle is used for the abscissa.

The case of cross-coupling due to the non-vanishing B_{16} and B_{26} was discussed by Reissner and Stavsky* for a two-layer angle-ply.

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* "Bending and Stretching of Certain Types of Heterogeneous Aeolotropic Elastic Plates," Journal of Appl. Mechanics, Paper 61-APM-21, 1961.

From Figure 5, it is clear that this coupling for a representative filament-wound composite is relatively weak. The coupling effect weakens very rapidly as n increases or θ deviates from 45°.

The D matrix for n even remains orhotropic. For n odd, D deviates markedly from orthotropic symmetry as can be seen from Figure 5 - the dimensionless D_{16} is 0.30 for n = 3. This means that for simple bending, the induced twisting moment is 30 percent of the induced bending moment. This is a very strong coupling and it does not decrease too rapidly as n increases. The ratio of $12\ D_{26}\ /\ h^2\ A_{22}$ is the same as the dimensionless D_{16} if the complement of the lamination angle is used.

SECTION 4

CONCLUSIONS

It is seen that the lamination of orthotropic unit plies can generate a number of interesting composite properties. Drastic variations can be made by changing the lamination parameters. Netting analysis, on the other hand, takes m into account only for the cross-ply and θ only for the angle-ply. The total number of plies, n, and the sequence of stacking are assumed to be immaterial. In an approximate manner, netting analysis takes into account the A matrix, but it agnores the B and D matrices.

Based on the continuum analysis, a number of materials optimization programs can be readily accomplished. Even for simple laminated plates and shells like the cross-ply and angle-ply, one has at his disposal sufficient lamination parameters to produce a wide range of desired composite properties. When this is coupled with the possible optimizations in the unit plies, for example, the unidirectionel filament-wound systems described in the First Quarterly Report of this contract,* a very wide range of controllable variations in composite materials is possible.

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*S. W. Tsai, "Structural Behavior of Composite Materials," First Quarterly Technical Report, Aeronutronic Publication #U-2122, May 24, 1963.

One interesting feature of laminated plates and shells is the transformation property of the composite material matrices A, B, and D. These matrices in general have different types of elastic symmetry from the symmetry of the original unit ply. If the unit plies are isotropic, with the properties of the odd plies different from those of the even plies, the distinction between cross-ply and angle-ply no longer exists. The A, B, and D matrices remain isotropic but they are independent. If the unit plies are orthotropic, as the case of unidirectional filament-wound systems, the resulting A, B, and D matrices may vary from the state of isotropy to complete anisotropy depending on the lamination parameters.

If one can produce a laminated plate of shell such that:

$$
\begin{aligned}
&A_{11} = A_{22}, \\
&A_{66} = (A_{11} - A_{12}) / 2 \\
&A_{16} = A_{26} = 0 \qquad\qquad (53) \\
&B_{ij} = 0 \\
\text{and} \quad &D_{ij} = h^3 A_{ij} / 12
\end{aligned}
$$

this laminated construction will behave as a homogeneous isotropic plate.* One can also build a laminated composite with other types of elastic symmetries, which can range from cubic, angular, and orthotropic

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*An application of this principle is being investigated by Mr. E. Scheyhing, Yale University, New Haven, Connecticut.

symmetries to completely anisotropic state. Or one can build a composite that has either gross homogeneity or varied degrees of heterogeneity. With a thorough understanding of the nature of the constitutive equation of laminated plates and shells a structural designer has at his disposal a new dimension in optimizing his design.

The experimental verification of the theoretical results of this report is in progress and will be presented in the future. The verification will be carried out in both plates and cylindrical pressure vessels.